WNE Linear Algebra Final Exam
Series A

1 February 2019

Please use separate sheets for different problems. Please provide the following data on each sheet

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Problem 1.
Let $V = \text{lin}((1,0,2,1),(1,1,0,2),(2,3,1,5),(0,1,-2,1))$ be a subspace of $\mathbb{R}^4$.

a) find a basis $\mathcal{A}$ of V and the dimension of V,
b) complete basis $\mathcal{A}$ to a basis $\mathcal{B}$ of $\mathbb{R}^4$ and find the coordinates of vector $w = (1,1,1,2) \in \mathbb{R}^4$ relative to $\mathcal{B}$.

Problem 2.
Let $V \subset \mathbb{R}^4$ be a subspace given by the homogeneous system of linear equations

$$\begin{cases} x_1 + 2x_2 + x_3 + 2x_4 = 0 \\ 3x_1 + 5x_2 + 4x_3 + 7x_4 = 0 \end{cases}$$

a) find a basis $\mathcal{A}$ and the dimension of the subspace V,
b) let $W \subset \mathbb{R}^4$ be a subspace spanned by the basis $\mathcal{A}$ and the vector $w = (1,0,0,0) \in \mathbb{R}^4$. Find a homogeneous system of linear equations which set of solutions is equal to W.

Problem 3.
Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear endomorphism given by the formula

$$\varphi((x_1,x_2,x_3)) = (x_2 + x_3, -2x_1 + 3x_2 + 4x_3, -x_3).$$

a) find the eigenvalues of φ and bases of the corresponding eigenspaces,
b) find a matrix $C \in M(3 \times 3; \mathbb{R})$ such that

$$C^{-1} M(\varphi)_{st}^{st} C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

for some $a \in \mathbb{R}$.

Problem 4.
Let $\mathcal{A} = ((1,2),(1,3))$, $\mathcal{B} = ((-1,-1),(0,1))$ be ordered bases of $\mathbb{R}^2$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformations given by the matrix

$$M(\varphi)_{\mathcal{B}}^{\mathcal{A}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a) find the formula of φ,
b) find the matrix $M(\varphi \circ \varphi)_{\mathcal{B}}^{\mathcal{B}}$.

Problem 5.
Let

$$A_t = \begin{bmatrix} 0 & 2 & t & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 \end{bmatrix}.$$

a) for which $t \in \mathbb{R}$ is matrix A_t invertible?
b) for $t = 2$ compute $\det(A_t B^{-1})$.

Problem 6.
Let $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -x_1 + x_2 + 2x_3 = 0\}$ be a subspace of $\mathbb{R}^3$.
a) find an orthonormal basis of V,
b) compute the orthogonal projection of $w = (1, 1, -3)$ onto $V^{\perp}$.

Problem 7.
Let $q_t \colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a quadratic form given by the formula $q_t((x_1, x_2, x_3)) = -x_1^2 - 4x_2^2 - x_3^2 + 2tx_1x_2$.
a) for which $t \in \mathbb{R}$ is the form q_t negative definite?
b) check if q_t is either positive semidefinite or negative semidefinite for $t = -2$.

Problem 8.
Consider the following linear programming problem $-5x_1 - x_3 - 6x_4 \to \min$ in the standard form with constraints

$$\begin{cases} x_1 + x_2 + 4x_4 + x_5 = 3 \\ 2x_1 + x_3 + 4x_4 - x_5 = 7 \end{cases} \quad \text{and } x_i \geqslant 0 \text{ for } i = 1, \ldots, 5$$

a) which of the sets $\mathcal{B}_1 = \{2, 3\}, \mathcal{B}_2 = \{2, 4\}$ is basic feasible? Write the corresponding feasible solution.
b) solve the linear programming problem using simplex method.